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A SOLVABLE NONLINEAR REACTION-DIFFUSION MODEL

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Abstract

We construct a coupled set of nonlinear reaction-diffusion equations which are exactly solvable. The model generalizes both the Burger equation and a Boltzman reaction equation recently introduced by Th. W. Ruijgrok and T. T. Wu.

Key-Words : non-linear dynamics, reaction-diffusion, solvable model.

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1 Introduction

Despite the enormous effort done since longtime, there are only few exactly solvable nonlinear field equations in $(1+1)$ dimensions. Among these models, the Burger equation is a well known example of diffusion nonlinear equation.

In their remarkable contribution [1], Th. W. Ruijgrok and T. T. Wu presented a completely solvable model of a discrete velocity, non-linear Boltzman equation (here referred as the RW equation). Basically, the non-linear evolution can be linearized via a logarithmic transformation of the Hopf-Cole type. In that sense, the RW model has been interpreted to be a generalization of the Burger equation [2]. Indeed, the Burger equation describes the evolution of the velocity field of a transport process governed by the White noise process (WNP), [3], while the RW equation can be viewed similarly as the evolution for the coupled velocity fields of a transport process governed by the random telegraph process, (RTP) [2].

In this paper we present a completely solvable nonlinear reaction-diffusion model. The starting point are the Chapman-Kolmogorof equations corresponding to a dynamical system driven by a sum of the WNP and the RTP. The nonlinear equations obtained after an inverse Hopf-Cole transformation are a couple set of nonlinear reaction-diffusion equations. The exact solution of this dynamics is expressible in terms of a linear differential operator acting on the logarithm of the convolution product of a Gaussian with the solution of the telegrapher's equation. In this way it is possible to construct all the solutions corresponding to initial conditions in the physical domain, i.e. such that they remain positive, as it might be for a distribution field. The model [1] and its solution is restituted when one of the control parameters (i.e. the diffusion constant governing the WNP) is set equal to zero.

2 The model and its solution

Consider the coupled set of non-linear reaction-diffusion equations :

$$\frac{\partial}{\partial t} + v \frac{\partial}{\partial x} f_1(x, t) = \mu \frac{\partial^2}{\partial x^2} f_1(x, t) + K_\mu(f_1, f_2, \frac{\partial}{\partial x} f_1, \frac{\partial}{\partial x} f_2) \quad (1)$$

$$\frac{\partial}{\partial t} - v \frac{\partial}{\partial x} f_2(x, t) = \mu \frac{\partial^2}{\partial x^2} f_2(x, t) - K_\mu(f_1, f_2, \frac{\partial}{\partial x} f_1, \frac{\partial}{\partial x} f_2) \quad (2)$$

with the non-linear operator K_μ defined as :

$$\begin{aligned} K_\mu = & -\alpha f_1 + \beta f_2 + f_1 f_2 \\ & + \frac{\mu S}{4v} \left[\frac{(\alpha+\beta)}{v} S - 4D_x + \frac{SD}{v} \right] \\ & + \mu^2 \left[-\frac{S^4}{16v^4} + \frac{1}{2} \frac{S^2 S_x}{v^3} - \frac{(S^2)_x}{2v^2} \right] \end{aligned} \quad (3)$$

where α, β, v, μ are positif reals, $S = f_1 + f_2$ and $D = f_1 - f_2$. Subscript x denotes the derivative with respect to x and the arguments (x, t) of f_1 and f_2 have been omitted in the definition of K_μ for simplicity.

Remark. When $\mu = 0$, the model defined by Eqs. (1), (2) and (3) reduces to the RW equation [1]. In this case the model has the following properties: (1) conservation of the number of particles and energy; (2) nonlinearity; (3) positivity of distribution functions and (4) uniqueness of the equilibrium distribution for any given density.

In our case diffusion clearly prevents property (1) and property (4) may only be verified with respect to each initial condition. The remaining properties are still verified in our case.

DEFINITION 2.1 *We define the following domain of admissible initial conditions:*

$$\begin{aligned} \Gamma(\alpha, \beta, \mu, v) = & \{f_1 \times f_2 \in C^2(\mathbf{R}) \times C^2(\mathbf{R}) | D(x, 0) \\ & - \frac{\mu}{v} S_x(x, 0) - \frac{\mu}{2v^2} S^2(x, 0) + \alpha + \beta\} \end{aligned} \quad (4)$$

We denote by $G(x, t)$ the usual density for the WNP, namely

$$G(x, t) = \frac{1}{\sqrt{2\pi\mu t}} \exp \left\{ -\frac{x^2}{\mu t} \right\} \quad (5)$$

and by $T(x, t)$ the corresponding density for the RTP, namely

$$T(x, t) = W(x, t) \exp \left\{ -\frac{\alpha + \beta}{2} t - \frac{\alpha - \beta}{2v} x \right\} \quad (6)$$

where

$$\begin{aligned}
W(x, t) = & \frac{1}{2} [W_0(x + vt) + W_0(x - vt)] + \\
& + \frac{1}{2} \int_{x-vt}^{x+vt} I_0 \left(\frac{\sqrt{\alpha\beta}}{v} (v^2 t^2 - (x - x')^2)^{\frac{1}{2}} \right) W_1(x') dx' \\
& + \frac{1}{2} \sqrt{\alpha\beta} t \int_{x-vt}^{x+vt} \left(\frac{I_1 \left(\frac{\sqrt{\alpha\beta}}{v} (v^2 t^2 - (x - x')^2)^{\frac{1}{2}} \right)}{\sqrt{v^2 t^2 - (x - x')^2}} W_0(x') \right) dx' \quad (7)
\end{aligned}$$

where I_0 and I_1 denote modified Bessel functions and

$$W_0(x) = \exp \left\{ -\frac{1}{2} \left[\int^x S(x', 0) dx' + \frac{\alpha - \beta}{v} \right] \right\} \quad (8)$$

and

$$W_1(x) = \frac{1}{2} \left\{ D(x, 0) - \frac{\mu}{\nu} S_x(x, 0) - \frac{\mu}{2\nu^2} S^2(x, 0) + \alpha + \beta \right\} W_0(x) \quad (9)$$

The next proposition is the main result of this paper.

PROPOSITION 2.2 *The solution of the set of reaction-diffusion equations (1), (2) for initial conditions $(f_1(x, 0), f_2(x, 0)) \in \Gamma$, reads*

$$f_1(x, t) = \mathcal{O}^{-\ln} (N(x, t)) \quad (10)$$

$$f_2(x, t) = \mathcal{O}^{+\ln} (N(x, t)) \quad (11)$$

with the operators \mathcal{O}^\pm defined by :

$$\mathcal{O}^\pm = \left(\frac{\partial}{\partial t} \pm v \frac{\partial}{\partial x} - \mu \frac{\partial^2}{\partial x^2} \right) \quad (12)$$

and

$$N(x, t) = \int_R G(x', t) T(x - x', t) dx' \quad (13)$$

PROOF We follow the idea of the construction presented in [1]. Hence, let the operators \mathcal{O}^\pm as defined in Eq.(12). These operators commute. Therefore it exists a field $F(x, t)$ such that Eqs. (1) and (2) are equivalent to :

$$\mathcal{O}^+ \mathcal{O}^- F(x, t) = -K_\mu \left(-\mathcal{O}^- F, \mathcal{O}^+ F, -\mathcal{O}^- \frac{\partial}{\partial x} F, \mathcal{O}^+ \frac{\partial}{\partial x} F \right) \quad (14)$$

with

$$-\mathcal{O}^- F(x, t) = f_1(x, t) \quad (15)$$

and

$$\mathcal{O}^+ F(x, t) = f_2(x, t) \quad (16)$$

Consider now the field $N(x, t) : (x, t) \in R \times R^+ \rightarrow R^+$ defined by :

$$N(x, t) = \exp\{-F(x, t)\} \quad (17)$$

Introducing Eq. (17) into Eq. (14), after some tedious manipulations, it is possible to verify that f_1, f_2 will satisfy the system (1) - (2) with K_μ defined as in (3) provided the field $N(x, t)$ obeys the following evolution equation :

$$\det \begin{pmatrix} -[\alpha + \mathcal{O}^+] & \beta \\ \alpha & -[\beta + \mathcal{O}^-] \end{pmatrix} N(x, t) = 0 \quad (18)$$

Eq. (18) is a fourth order hyperbolic partial differential equation with a probabilistic interpretation, [4, 5]. Consequently, Eq. (18) conserves the positivity of its solution. This equation has been first encountered in [4] (see Eq. (21) of this reference) and its solution is given in [5], where it is shown that :

$$N(x, t) = (G * T)(x, t) \quad (19)$$

with $*$ denoting the convolution product as in Eq.(13) with :

$$\frac{\partial}{\partial t} G(x, t) = \mu^2 \frac{\partial^2}{\partial x^2} G(x, t) \quad (20)$$

and

$$\left(\frac{\partial^2}{\partial t^2} + (\alpha + \beta) \frac{\partial}{\partial t} \right) T(x, t) = \left(v^2 \frac{\partial^2}{\partial x^2} + v(\alpha - \beta) \frac{\partial}{\partial x} \right) T(x, t). \quad (21)$$

From Eqs. (10), (11) and (12), we have :

$$N(x, 0) = N_0(x) = \exp\left\{-\frac{1}{2v} \int dx (f_1(x, 0) + f_2(x, 0))\right\} \quad (22)$$

and

$$\frac{\partial}{\partial t} N(x, 0) = N_1(x) = \frac{1}{2} \{f_1(x, 0) - f_2(x, 0) + \frac{2\mu}{v} \frac{\partial}{\partial x} [f_1(x, 0) + f_2(x, 0)]\} N_0(x). \quad (23)$$

In order to get $G(x, t)$ as in Eq.(5) it is sufficient to start with the initial value in the form :

$$G(x, 0) = \delta(x) \quad (24)$$

Now, concerning $T(x, t)$, in order to match Eq. (6.4) of [1] we introduce $W(x, t)$ defined by Eq. (6). In view of (19) and (6) $N(x, t)$ will be known as far as the solution $W(x, t)$ of the following equation is found :

$$\frac{\partial^2}{\partial t^2} W(x, t) = \left(v^2 \frac{\partial^2}{\partial x^2} + \alpha \beta \right) W(x, t) \quad (25)$$

To integrate (25) one needs initial conditions. The last are related to the initial conditions for $N(x, t)$ by

$$W(x, 0) = W_0(x) = \exp\left\{\left(\frac{\alpha - \beta}{2v}\right)x\right\} N_0(x) \quad (26)$$

and hence with the initial conditions for f_1 and f_2 . Therefore they may be computed using (19)-(24) which leads to (8) and (9).

Finally, thanks to Eqs.(8), (9) and (26), the solution of Eq. (25) is given in [1] and reads as written in Eq. (7). For the initial values $(f_1(x, 0), f_2(x, 0)) \in \Gamma$ as defined by Eq. (4), $W_0(x)$ and $W_1(x)$ are positive. Hence $W(x, t) \geq 0, \forall (x, t) \in R \times R^+$, see Lemma 1.3.2 in [6]. Hence $N(x, t)$ is positive since by Eq.(19) it is the convolution of two positive fields. Therefore $\ln(N(x, t))$ is well defined. The proof is complete.

3 Concluding remarks

In this note, we have shown how to extent the exactly solvable discrete velocity non-linear Boltzman equation introduced by Th. W. Rujgrok and T. T . Wu [1], (the RW equation). In this way we obtain a nonlinear reaction-diffusion coupled system of equations for which the exact solutions can be computed as far as the corresponding initial conditions are in the positivity preserving domain. Our construction is based on the observation that that the RW equation and the Burger equation are non-linear evolutions for velocity fields of linear transport processes, a point of view pioneered in [4] and further extended in [2]. Accordingly, the construction presented here can

be further generalized by considering sets of non-linear equations for velocity fields describing a transport process with dynamics governed by a mixed noise stochastic differential equations [5]. Therefore, the model presented here is only one of the simplest of a whole family to be further investigated.

From that perspective, the special case sorted out here might appear to be an arbitrary choice. However, due to the simple form of its solutions and its reaction-diffusion form, it may deserve special attention.

In a forthcoming paper we shall study some properties of the solutions of this model.

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